Cleve's Corner: *The World's Simplest Impossible Problem*

The other day at lunch with a couple of other MathWorks people, I posed the following problem:

"I'm thinking of two numbers. Their average is 3. What are the numbers?"

Now stop a minute and, before you read any further, provide your own answer to my question.

Of course, my problem doesn't have a "right" answer. The solution is not unique. The problem is "underdetermined" or "ill-posed."

Most of the people I was having lunch with wouldn't give me an answer. When I pressed them, one person finally said, "Both numbers could be 3." Another person said, "Yes, but one could be 6 and the other 0." That's what I had hoped they would say. In some sense, these are both "nice" answers. Nobody said "23 and -17," or "2.71828 and 3.28172." Those would have also been correct, but not as "nice."

What does this have to do with MATLAB? Well, MATLAB doesn't balk at impossible problems. It will solve this one. And it does it without complaining like my buddies at lunch.

Naturally, the problem is a matrix problem. It is a "system"

```
A*x = b
```

of one "simultaneous" linear equation in two unknowns. The matrix is

```
A = [1/2 1/2]
```

and the right-hand side is

```
b = 3
```

MATLAB's backslash solves such equations. Typing

```
x = A\b
```

tells me

```
x =
      6
      0
```

The solution is a 2-by-1 matrix representation of one of the "nice" answers I expected.

The second half of the `help` entry for "\" gives some indication where this solution came from.

```
If A is an m-by-n matrix with m <
or > n and B is a column vector
with m components, or a matrix
with several such columns, then X
= A\B is the solution in the least
squares sense to the under- or
overdetermined system of equa-
tions A*X = B.  The effective
rank, k, of A is determined
from the QR decomposition with
pivoting.  A solution X is
computed which has at most k
nonzero components per column.
If k < n this will usually not
be the same solution as
PINV(A)*B.
```

In our case, we have `m = 1` and `n = 2`. My matrix has full rank, but the most that can be is `k = 1`. So, we get a solution with at most one nonzero component. Even that isn't unique. There are exactly two solutions with one nonzero component, `[6 0]'` and `[0 6]'`. We get the one where the nonzero component has the smallest index.

What about my other "nice" solution? That comes from the pseudoinverse.

```
x = pinv(A)*b
```

produces

```
x =
      3.0000
      3.0000
```

(The fact that I got `3.0000` instead of the integer `3` without the trailing zeros indicates that there has been some roundoff error and I don't have my other "nice" solution exactly, but you can pursue that yourself if you really want to.)

Of all the possible solutions to `A*x = b`, this one, `x = [3 3]'`, is the "shortest." It minimizes `norm(x)`. In fact `x = A\b` has

```
norm(x) = 6.0000
```

while `x = pinv(A)*b` has

```
norm(x) = 4.2426
```

Now, finally, we have uniqueness. Of all possible solutions to `A*x = b`, the one which also minimizes `norm(x)` is unique.

So, MATLAB not only solves the problem, it gives us a choice between two different solutions, `x = A\b` and `x = pinv(A)*b`. I think the first solution, `A\b = [6 0]'`, is "nice" because it is "simple"; i.e., it has the fewest possible nonzero components. I think the second solution, `pinv(A)*b = [3 3]'`, is "nice" because it is "short" and "unique." None of the other possible solutions have such nice characterizations.

This problem, "Given the average of two numbers, find the numbers," captures the essence of many ill-posed and underdetermined problems. Com-

Cleve's Corner *continued on next page*

Cleve's Corner *continued from previous page*

puter tomography, which is the life-saving business of generating images from X-ray, magnetic resonance, and other scanners, is really a grown-up version of this question. Additional constraints, like minimum norm or fewest nonzero components or "good looking picture," have to be specified to make it a reasonable mathematical and computational task.

By the way, I first learned about this "World's Simplest Impossible Problem" from Don Morrison, who also started the University of New Mexico's Computer Science Department , invented the Fast Fourier Transform before Cooley and Tukey, and, years ago, got me to move to New Mexico. Thanks for all those things, Don.

I have to confess that I wrote this anecdote about posing a problem at a MathWorks lunch before it really happened. The results of the actual experiment were even more interesting. As I had expected, everybody did grumble and complain about my problem. Then one person said "9 and -3." She had obviously picked up on the idea. Another person said "3 and 1." Luckily, he is not responsible for any of the numeric portions of MATLAB. But three other people all said "2 and 4." That is certainly another "nice" answer, but the constraints it satisfies are more subtle. They have something to do with requiring the solution to have integer components that are distinct, but near the specified average. It's harder to state and compute the solution in MATLAB without just giving the answer.

Okay, now I'm thinking of three numbers whose average is π. What are the numbers?

Cleve Moler is Chairman and co-founder of The MathWorks, Inc. If you have any comments regarding the material discussed in "Cleve's Corner," you may contact Cleve via email at `moler@mathworks.com`.

Papers Received *continued from page 4*

MATLAB as a high-level programming language for teaching and learning multivariate chemometric procedures. Examples are given of applications to multiwavelength spectrophotometry, iterative least-squares curvefitting by simplex optimization, evolving factor analysis, and rank annihilation.

Theoretical Limitations to Parallel Processing of Matrix Multiplication Algorithms and Hardware Accelerators, by Roland Cooke (University of Idaho). This paper develops the theoretical limitations to parallelism and the resultant performance bounds of vector times scalar, vector times vector, and matrix times matrix multiplication for a single bus environment. Bounding equations derived for vector times scalar and vector times vector show that hardware accelerators will not significantly improve the performance of these operations over available LINPACK Basic Linear Algebra Subroutines (BLAS).

The Angle of the Transverse Force Acting on a High-Speed V-Bottom Hull, by I. Boguslavsky (Cameri), A. Biran (Technion), and D. Rosen (Cameri), from the 23rd Israel Conference on Mechanical Engineering, Technion City, Haifa, Israel, May 21 - 22, 1990. This report describes research being carried on by the Coastal and Marine Engineering Research Institute, in which MATLAB was used for evaluating and plotting the authors' data.

An On-board Control System for Mobile Robots, by A. Micaelli, P. Mandin, C. Tahmi, L. Boissier, and J.M. Detriche (CEA, Unité de Génie Robotique Avancé). The authors present the architecture of an on-board control system for mobile robots, which was developed in the Advanced Robotics Engineering Unit of the Commissariat for Atomic Energy at the Center for Nuclear Studies in Fontenay-aux-Roses, France.